# Approximated Cycles in TU-Games

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**Abstract.** Non-balanced $TU$-games, which are games with empty core, can also be characterized by the existence of certain type of cycles of pre-imputations. A particular sub-class of these cycles is that of maximal $U$-cycles. They show up in connection with the application of a transfer scheme to games with empty core, although this transfer scheme was originally designed however, to reach a point in the core of a balanced $TU$-game. While the appearance of one of such cycles is enough to detect the non-balancedness of the game, it is still an open question if every non-balanced game has a maximal $U$-cycle. The aim of this note is to show, however, that there are always, in non-balanced games, approximated maximal $U$-cycles sharing with the true maximal $U$-cycles all their good properties.

## 1 Introduction

The core of a game is one of the most widely used concept of solution in cooperative game theory. Bondareva ([1]) and Shapley ([5]) characterize those game with transferable utility ( $TU$-game) having non-empty core as the class of balanced games. The key concept here is that of balanced family of coalitions (Section 2). More recently, Cesco ([2],[4]) gave two more characterization of balanced games, although in this case, the key notion to prove the results is that of cycle of pre-imputations. The results state that a $TU$-game has empty core if and only if certain type of cycles, fundamental cycles in the first reference, $U$-cycles (Section 2) in the second one, are present in the game.

A $U$-cycle is a particular version of a fundamental cycle and appears in connection with a transfer scheme introduced by [2] to reach a point in the core of a balanced $TU$-game. A transfer scheme is a bargaining dynamic processes, consistent with the standards of rational behavior, leading from an unacceptable payoff to an acceptable one. The sequences generated by the transfer scheme developed in [2] are called maximal $U$-sequences, and it is proven there that they always converge to an imputation in the core of a $TU$-game with empty core. On the other hand, a numerical algorithm to generate maximal $U$-sequences has been developed. When it was used in the framework of non-balanced games, the maximal $U$-sequences generated by the algorithm have not converged at all as was expected. However, all the numerical examples worked out in this sub-class of games have revealed the numerical appearance of limit cycles of pre-imputations, in the sense of dynamical systems. This behavior suggests that maximal $U$-sequences should "converge" to such limit cycles. That numerical observed pattern is plausible since, as we will show below, the set of pre-imputations in a maximal $U$-sequence is always a bounded set. Nevertheless, no general convergence theoretic result has been proven yet, even though the boundedness of maximal $U$-sequences justify the appearance of approximate maximal $U$-cycles instead. The aim of this note is to present some results showing that these approximated cycles, which are numerically accessible, are able to provide the same kind of information that true cycles provide. For instance, we will prove that a family of coalitions, naturally related to them, is balanced, like in the case of maximal $U$-cycles. We will also prove that non-balanced games can be characterized by the appearance of approximated cycles, paralleling thus, the main results proven by [3] and [4]. We also present in this note a result indicating how good the approximation has to be in order for an approximated maximal $U$-cycle give the same information as a maximal true $U$-cycles provide.

## 2 Preliminaries

A *TU-game* is an ordered pair $(N, v)$, where $N = \{1, 2, ..., n\}$ is a finite non-empty set, the set of *players*, $v$ is the *characteristic function*, which is a real valued function defined on the family of subsets of $N, \mathcal{P}(N)$ satisfying $v(\phi) = 0$. The elements in $\mathcal{P}(N)$ are called *coalitions*. The set of *pre-imputations* is $E = \{x = (x_1, \ldots, x_n) \in R^n : \sum_{i \in N} x_i = v(N)\}$, and the set of *imputations* is $A = \{x \in E : x_i \geq v(i)$ for all $i \in N\}$.Given a pre-imputation $x$ and a coalition $S$ in a game $(N, v)$, the *excess* of the coalition $S$ with respect to $x$ is $e(S, x) = v(S) - x(S)$, where $x(S) = \sum_{i \in S} x_i$ if $S \neq \phi$ and 0 otherwise. The *core* of $(N, v)$ is the set $C = \{x \in E : e(S, x) \leq 0$ for all $S \in \mathcal{P}(N)\}$.

The core of a game may be the empty set. The Shapley-Bondareva theorem ([1], [5]) characterizes the sub-class of $TU$-games with non-empty core, where the notion of balanced subfamily of coalitions plays a central role. A non-empty family $\mathcal{B}$ is *balanced* if and only if there exists a set of positive real numbers $(\lambda_S)_{S \in \mathcal{B}}$, the *balancing weights*, such that $\sum\limits_{S \in \mathcal{B}(i)} \lambda_S = 1$ for all $i \in N$. Here $\mathcal{B}(i) = \{S \in B : i \in S\}$ . A game $(N, v)$ is balanced if $w(\mathcal{B}, v) = \sum_{S \in \mathcal{B}} \lambda_S v(S) \leq v(N)$ for all balanced family $\mathcal{B}$ with balancing weights $(\lambda_S)$. Shapley- Bondareva's theorem states that the core of a $TU-$game is non-empty if and only if the game is balanced.

A *minimal* balanced family is one that includes no other proper balanced subfamily, and it has a unique set of balancing weights ([5]).

Balancedness can also be defined as follows. Let $\chi_S \in R^n$ denotes the vector defined by $(\chi_S)_i = 1$ if $i \in S$ and 0 otherwise (the indicator vector of $S$). Then, a family $\mathcal{B}$ is balanced if there exist positive balancing weights $(\lambda_S)_{S \in \mathcal{B}}$ such that $\sum_{S \in \mathcal{B}} \lambda_S \chi_S = \chi_N$. For each non-empty, proper coalition $S \in \mathcal{P}(N)$, let us define

$$\beta_S = \frac{1}{|S|}\chi_S - \frac{1}{|N\backslash S|}\chi_{N\backslash S}.$$

Given a non-empty family $\mathcal{B}$ of proper coalitions of $N$, $B_{\mathcal{B}}$ will stand for the matrix having as its columns the vectors $\beta_S, S \in \mathcal{B}$. Then the following characterization for balanced families of coalitions holds ([3], Theorem 6).

**Theorem 1.** *A family $\mathcal{B}$ of non-empty proper coalitions of $N$ is balanced if and only if the linear system*

$$B_{\mathcal{B}}.y = 0 \tag{1}$$

*has a solution $y = (y_S)_{S\in\mathcal{B}}$ with $y_S > 0$ for all $S \in \mathcal{B}$ (positive solution).*

A *transfer sequence* in a game $(N, v)$ is a sequence of pre-imputations $(x^k), k \geq 1$, such that there exist associated sequences of positive real numbers $(\mu_k), k \geq 1$ and $(S_k), k \geq 1$ of non-empty, proper coalitions of $N$ (not necessarily all different) satisfying the neighboring transfer properties

$$x^{k+1} = x^k + \mu_k.\beta_{S_k} forallk \geq 1.$$

A *cycle* **c** is a finite transfer sequence $(x^k)_{k=1}^{m+1}, m \geq 1$, such that

$$x^{m+1} = x^1, \tag{2}$$

and

$$\mu_{m+1} = \mu_1, S_{m+1} = S_1,$$

as well.
A cycle is *fundamental* if $\mu_k \leq e(S_k, x^k)$ for all $k = 1, ..., m$.
A cycle is a $U$-*cycle* if $\mu_k = e(S_k, x^k)$ for all $k = 1, ..., m$.
A $U$-cycle is *maximal* if for all $k = 1, ..., m, e(S_k, x^k) \geq e(S, x^k)$ for all coalition $S$. Given a cycle $\mathbf{c} = (x^k)_{k=1}^m$, we denote the vector of coalitions $(S_k)_{k=1}^m$ by supp($\mathbf{c}$) and the entries of the vector $(\mu_k)_{k=1}^m$ will be referred to as the transfer amounts. Let $\mathcal{B}(\mathbf{c}) = \{S : S = S_k$ for some entry of supp($\mathbf{c}$)$\}$. We will refer to $\mathcal{B}(\mathbf{c})$ as the family of coalitions supporting **c**, and unlike supp($\mathbf{c}$), all its members are different. Given a fundamental cycle **c,** the coalitions in supp($\mathbf{c}$) and $\mathcal{B}(\mathbf{c})$ are both balanced families ([3], Theorem 1).

The existence of cycles in a $TU$-game is strongly related to the non-existence of points in the core of the game. The following characterization result is derived from the main results by Cesco ([3],[4]).

**Theorem 2.** *Let $(N, v)$ be a $TU$-game. Then, the following statements are equivalent.*

*a) The core of $(N, v)$ is empty.*
*b) There exists a fundamental cycle in $(N, v)$.*
*c) There exists a $U$-cycle in $(N, v)$.*

The key to prove that $b$) implies $a$) is a good representation for the worth $w(\mathcal{B}, v)$ of supp($\mathbf{c}$) with respect to a very specific set of balancing weights determined in terms of information gathered from the cycle.

If in (2) above we replace the equality between $x^{m+1}$ and $x^1$ by

$$x^{m+1} - x^1 = b^\varepsilon$$

with $\|b^\varepsilon\|_2 \leq \varepsilon$, (whatever $\mu_{m+1}$ and $S_{m+1}$ could be), we will have an *$\varepsilon$-cycle*. Similarly to cycles, we will also speak about fundamental, $U$ and maximal $U$ $\varepsilon$-cycles.

## 3 Results

It would be interesting to be able to add the claim "$d$) There exists a maximal $U$-cycle in $(N, v)$" to Theorem 2 above. Clearly the existence of a maximal $U$-cycle implies the emptiness of the core of the game, since the maximal $U$-transfer scheme starting at any pre-imputation of the cycle would never converge, as it should be, if the core were non-empty ([2]). However, we have not been capable yet to prove that, whenever a game has empty core, there is a maximal $U$-cycle, and we can only report partial results. Besides the existence problem, there is another important issue to be dealt with, namely, if there exists a computational device to generate such kind of cycles. Some computational experience based on an algorithm implementing the transfer scheme introduced by [2], when applied to games with non-empty core, shows that the sequences generated by the algorithm (maximal $U$-sequences) have always limit cycles of pre-imputations (in the sense of dynamical systems) which are, in fact, numerical maximal $U$-cycles (maximal $U$-cycles within the machine tolerance). We do not have a formal proof yet that every maximal $U$-sequence has a "true" maximal $U$-cycle as a limit cycle. However, the next result justifies this observed behavior. It states that maximal $U$-sequences are bounded.

**Theorem 3.** *Let $(N, v)$ be a TU-game. Then, every maximal $U$-sequence in $(N, v)$ is bounded. Moreover, any maximal transfer sequence starting at a pre-imputation $x^1$ will be bounded (in the 2-norm) by* $\max\left\{\left\|x^1\right\|_2, \sqrt{K_1 + K_2}\right\}$, *where $K_1$ and $K_2$ are positive constants independent of $x^1$.*

We will refer to the associated family of coalitions $\mathcal{B} = (S_k)_{k=1}^m$ as the supporting family of the $\varepsilon$-cycle. If certain conditions prevail, this family is balanced as it is shown in the next result.

**Theorem 4.** *Let $(N, v)$ be a TU-game and let $(x^k)_{k=1}^{m+1}$ be a $\varepsilon$-cycle of pre-imputations such that $e(S_k, x^k) \geq \delta$ for all $k = 1, ..., m$ for some positive $\delta$. Then, if $\varepsilon \leq \frac{1}{2K(n)}\delta$, the supporting family $\mathcal{B} = (S_k)_{k=1}^m$ is balanced.*

To prove this theorem, the following auxiliary result plays a key role.

**Lemma 1.** *Let $\mathcal{B}$ a family of $m$ different proper coalitions of $N$. Let us assume that there exists a positive solution $y^{\varepsilon}$ for the linear system*

$$B_{\mathcal{B}}.y^{\varepsilon} = b^{\varepsilon}$$

*with $0 < \|b^{\varepsilon}\|_2 \leq \varepsilon, \varepsilon > 0$. Then there is a solution $\hat{x}$ of the homogeneous system $B_{\mathcal{B}}.x = 0$ such that $\|\hat{x} - y^{\varepsilon}\| \leq K(n).\varepsilon$ where $K(n)$ is a constant depending on $n$ solely.*

*Note 1.* The constant $K(n)$ is related somehow with the condition number of the $B_{\mathcal{B}}$ matrices. On the other hand, the positive number $\delta$ can be chosen depending on the game $(N, v)$ solely.

The next two propositions will characterize games with empty core in terms of maximal $\varepsilon$-cycles.

**Proposition 1.** *Let $(N, v)$ be a TU-game and $\mathbf{c} = (x^k)_{k=1}^{m+1}$ an $\varepsilon$-cycle of pre-imputations with $e(S_k, x^k) \geq \delta > 0$ for all coalition $S_k$ belonging to its associated supporting family of coalitions $\mathcal{B} = (S_k)_{k=1}^{m}$, for some $\varepsilon \leq \frac{1}{4n^3 K(n)}\delta$. Then the game is non-balanced.*

Of course, the $\varepsilon$-cycles in the above result can be maximal $\varepsilon$-cycles. In this case, a converse of Proposition 1 also holds.

**Proposition 2.** *Let $(N, v)$ be a non-balanced game. Then there exist $\delta > 0$, $0 < \varepsilon \leq \frac{1}{4n^3 K(n)}\delta$ and a maximal $\varepsilon$-cycle $(x^k)_{k=1}^{m+1}$ of pre-imputations with $e(S_k, x^k) \geq \delta$ for each coalition $S_k$ belonging to its supporting family of coalitions $\mathcal{B} = (S_k)_{k=1}^{m}$.*

**Theorem 5.** *Let $(N, v)$ be a game. Then, it is non-balanced if and only if there exists a maximal $\varepsilon$-cycle $(x^k)_{k=1}^{m+1}$ satisfying $e(S_k, x^k) \geq \delta$ for some $\delta > 0$ and for each coalition $S_k$ belonging to its supporting family of coalitions $\mathcal{B} = (S_k)_{k=1}^{m}$ while $0 < \varepsilon \leq \frac{1}{4n^3 K(n)}\delta$.*

## 4 Conclusions

In this note we develop an approximate theory to maximal $U$-cycles. We have available an algorithm to get an approximate cycle whose extreme points (initial and final points) are as close as we wish. Moreover, if the distance between this extreme points is less or equal than a constant depending of the number of players and a bound for the excesses related to the imputations in the approximate cycle, then we can assure that its supporting family of coalitions is balanced, and moreover, that the game is non balanced. Thus, we have a practical numerical device to get approximate maximal $U$-cycles capable to provide the same information that true maximal $U$-cycles do. It is worth noting that in all the examples worked out, the supporting family of the cycles obtained by the algorithm has been a minimal balanced family.

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